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Difficulties With Using Continuous Fractal Theory for Discontinuity Surfaces

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Roughness of discontinuities and relations between shear strength and roughness are of major interest in rock mechanics. Options to characterize roughness of a discontinuity, in terms of a fractal and/or fractal dimension, are investigated mathematically and by simulated roughness profiles. The results show that relations between the JRC roughness profiles and fractal dimension determined by the divider method do not prove that the JRC curves are fractals, can easily be inaccurate and can lead to erroneous relations. If roughness profiles are fractals at all, then the divider method is likely not suitable for determining the fractal dimension.

INTRODUCTION

Roughness of a discontinuity is a major factor determining the shear strength along discontinuities. In particular, roughness determines the shear strength in situations for which the normal stress on the discontinuity is relatively small compared to the discontinuity wall strength [4] preventing shearing through asperities. Describing the roughness of a discontinuity by a mathematical expression that relates roughness to shear strength has been attempted by various authors [1–3, 9, 15, 17]. Barton and Choubey [4] introduced the JRC curves and defined an empirical relation between the JRC number and shear strength. However, in practice it is often difficult to determine the proper JRC number for a given discontinuity surface.

Several authors [7, 8, 12, 17, 19] have tried to use the results of fractal theory in describing discontinuity surfaces. They found a good correlation between the JRC number and the fractal dimension and concluded that the JRC curves can be represented by a fractal dimension. In this paper, the fractal dimension of the JRC curves is determined with the divider method and it has been found that the logarithm of the number of divider intervals of size ϵ ($\ln N(\epsilon)$) versus the logarithm of the scale parameter ($\ln 1/\epsilon$) does not show a straight line. The minimum condition although not conclusive, to prove that a curve is a fractal is a straight line for this method of determination [10, 13, 14]. In this case, the surface might be a real fractal and is scale independent according to the definition of the continuous fractal theory [11, 14, 16]. As no straight line has been found, it is doubtful that the JRC curves are fractal.

The fractal dimension is based on using continuous fractal theory on discrete data. The data sets used are inevitably discrete for practical reasons. There is always a limit to the amount of data points available, whereas fractal theory assumes an unlimited amount of data points [11].

The implications of using continuous fractal theory on discrete data sets are therefore investigated. Fractal data sets with a known fractal dimension have been generated. The fractal dimensions of the data sets have been determined using the divider algorithm [11] (termed the divider dimension in this paper). This allows for study of the effects caused by changing a continuous curve into a discrete data set. The resulting divider dimension should theoretically be the same as the fractal dimension of the original data set. However, it was found that the fractal dimension calculated by the divider method differs from the original fractal dimension depending on the amount of discrete points and on the amplitude (half the vertical range) of the data set. This effect diminishes with a higher density of discrete points for larger amplitudes. To verify the theoretical predictions, the dimension of a sine function, which has the fractal dimension 1.0, was tested. Finally the results of the previous studies [7, 12, 19] on the fractal dimension of the

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JRC curves have been evaluated and the fractal dimensions are determined after increasing the number of data points.

DIVIDER ALGORITHM

The divider dimension D of a curve F in $\mathbb{R}^2$ is defined as:

$$D = \limsup_{\epsilon \downarrow 0} \frac{\ln N(\epsilon)}{\ln(1/\epsilon)} \tag{1}$$

with ϵ a scale parameter and $N(\epsilon)$ the number of scale lengths ϵ needed to cover F [11].

The dimension of F will be constant over all values of the scale parameter ϵ if the curve F is self-similar. This also implies that, for a self-similar curve the number $N(\epsilon)$ follows a power law of the form:

$$N(\epsilon) \propto \epsilon^{-D} \tag{2}$$

The plot of $\ln 1/\epsilon$ versus $\ln N(\epsilon)$ for $\epsilon > 0$ will be a straight line with tangent equal to D. Thus, for a self-similar curve the limit in equation (1) need not be evaluated. It is sufficient to calculate $\ln N(\epsilon)$ as a function of $\ln 1/\epsilon$ for relevant values of ϵ and determine the slope of the line through these points. It is in this way that the divider algorithm is used to reconstruct the dimension of a curve from a sample set.

Now consider a data record similar to the profile sampling of a rough surface. Let S be a sample $\{(x_k, y_k), k = 1, \ldots, K\}$ (resolution limits imposed by the number of samples). To find the dimension of S, a curve F is constructed consisting of all points (x_k, y_k) and linearly interpolated in between (x_{k+1}, y_{k+1}) and (x_k, y_k), $k = 1, \ldots, K-1$. The curve F is not self-similar at all scales of ϵ because of the linear interpolation. However, if S is sampled from a self-similar curve, on some interval I_ϵ of ϵ, the graph of $(\ln 1/\epsilon, \ln N(\epsilon))$ will appear to be straight. The estimated tangent of this straight segment is the divider dimension.

The divider algorithm is implemented as follows. Calculate $N(\epsilon)$ for some value of ϵ, determine, by visual interpretation of the graph $(\ln 1/\epsilon, \ln N(\epsilon))$, the range where the graph is linear and estimate the tangent of this segment. Since on this linear segment the data can show some scatter, linear regression is applied on the data points $(\ln 1/\epsilon, \ln N(\epsilon))$ of this segment [6, 10].

LIMITS ON THE DIVIDER ALGORITHM

It is known that some non-fractal forms can show fractality when analysed by the divider method [10]. In this paper, the fractal dimension is analysed by the divider method for a discrete data set sampled from a true fractal.

Analysis of the fractal dimension by the divider method of a discrete data set is limited by the range for the scale parameter (ϵ). For real discrete data sets the lower limit of ϵ is determined by the interval between the data points. For sampled data this limit is the information limit (ϵ_0), below which the sampled points belong to straight line segments. The upper limit of ϵ is determined by the length of the data set. Only within those limits it may be possible to trace back a power law behaviour consistent with the actual fractal dimension. However, two effects influence the calculated divider dimension.

The first effect makes the divider dimension smaller than the true fractal dimension, D, for $D > 1$. The divider algorithm assumes that the data set is linear for ϵ below the sample resolution limit ϵ_0. Suppose F is a set of samples (x, y) from a data set with fractal dimension D ($D > 1$). Assume also that the sample interval ϵ_0 is above the information limit. Let $N(\epsilon)$ be the number of line segments with length ϵ needed to cover F. For $\epsilon \geqslant \epsilon_0$, $N(\epsilon) \propto \epsilon^{-D}$. Therefore, $N(\epsilon_0) = \alpha \cdot \epsilon_0^{-D}$ in which α does not change with scale.

At a scale below ϵ_0 the curve of F is piece-wise linear consisting of line segments with length ϵ_0. Let $\epsilon_1 = M^{-1}\epsilon_0$ where M is an integer greater than 1. To cover a line segment ϵ_0 only M line segments of length ϵ_1 are needed. So $N(\epsilon_1) = \alpha M \epsilon_0^{-D}$, hence $N(\epsilon_1)$ increases linearly with M. Thus at a scale below the sample resolution limit (and/or below the information limit) the power law behaviour is linear corresponding to a dimension of 1.0. By applying the divider algorithm and deriving the divider dimension of the set $\{(\ln 1/\epsilon, \ln N(\epsilon))\}$, a slope between 1 and D will be found. In general, the divider algorithm underestimates the true fractal dimension D ($D > 1$) if the sampled set is finite. The graph of a finite data set is a broken, finite, piece-wise linear curve with no infinite self-similarity. On the linear parts, the power law behaviour corresponds to a dimension of 1.0 while the assumed fractal dimension is greater than 1.0. Consequently the calculated divider dimension will be lower than the assumed fractal dimension.

The second effect results in overestimating the assumed fractal dimension D if $D = 1$. Suppose the graph of $\{(x, y)\}$ is a continuous curve with finite length and a true dimension of 1.0. The length (L) is approximated by the divider method as $N(\epsilon) \cdot \epsilon$, where:

$$\lim_{\epsilon \downarrow 0} (N(\epsilon) * \epsilon) = L \tag{3}$$

Equivalently, as $\epsilon \downarrow 0$ then $\ln N(\epsilon) + \ln \epsilon \rightarrow \ln L$; since L is a constant, by dividing $\ln(1/\epsilon)$ it follows:

$$\lim_{\epsilon \downarrow 0} \frac{\ln N(\epsilon)}{\ln(1/\epsilon)} = 1 \tag{4}$$

Thus the divider dimension equals the true fractal dimension. However, with finite data sets the limit $\epsilon \downarrow 0$ cannot be calculated and the divider algorithm has to be applied. Suppose the data set $\{(x, y)\}$ forms a piece-wise linear but broken curve, which in total is not part of a straight line. Then the linear self-similarity is lost because in the neighbourhood of the breakpoints the power law is non-linear. Although only finite breakpoints may exist they cannot be cancelled out, since the limit $\epsilon = 0$ cannot be reached. Hence the divider algorithm will result in a dimension greater than 1.0 corresponding to the non-linearity at the breakpoints.

This argument is supported by the construction of non-fractal curves with a divider dimension greater than 1.0 in Dutch [10].

As a consequence of both effects the divider algorithm cannot discriminate between finite sets of low fractal dimension and finite sets with a fractal dimension of one. This is confirmed by the following experiments.

TEST PROCEDURES

Fractal data sets have been created using the procedure described by Barnsley [5] [Fig. 5(a) and (b)]. This procedure creates a discrete data set with a given fractal dimension. The amount of discrete points along the fractal, the length and the amplitude of the initial profile used for this procedure can be freely chosen. A total of 180 data sets has been generated with different fractal dimension, amplitude and amount of discrete points along the fractal. The data sets have a length of 0.1 m and an amplitude (measured after creating the fractal profiles) increasing from 0.001 to about 0.038 m. the length and the amplitude are in the same order as those of the JRC curves. The fractal data set is described by 200, 400, 800 and 1600 points. The fractal dimensions range from 1.0 to 1.4.

GENERATED FRACTAL CURVES

Figure 1 shows the divider dimensions determined for fractal data sets with different fractal dimensions and amplitudes. The amount of discrete points along the fractal curves are 200, 400, 800 and 1600 points for Fig. 1(a), (b), (c) and (d), respectively. The theoretical result should be a divider dimension independent of the amplitude and equal to the fractal dimension of the data set. However, clearly the divider dimension depends on the amplitude of the data set. Increasing the number of points from 200 to 1600 does not improve the results for the smaller amplitudes. For higher amplitudes the divider dimension does approximate the fractal dimension.

Figure 2 shows as an example, the logarithmic graph for the data set with a fractal dimension of 1.4, an amplitude of 0.012 m and consists of 200 sample points over a length of 0.1 m. The line shown is not perfectly straight, which it should be for a strict fractal. The irregular behaviour becomes more obvious by plotting the first differences ($\delta \ln N(\epsilon)$) (Fig. 3).

SINE CURVES

The effects caused by the data point density and amplitude of the data set on the determination of the divider dimension prompted an investigation of the divider dimension for theoretical exactly-known cases as also discussed by Maerz and Franklin [13].

The sine function is a continuous function with a dimension of 1.0. The sine function is investigated with the same procedure as the generated fractal curves and with amplitudes in the same order as the generated fractal data sets. The function is sampled 200 and 400 times over a length of 0.1 m. The calculated divider dimension (Fig. 4) shows the same effects as for the generated fractal data sets. The divider dimension increases with increasing amplitude.

JRC PROFILES

The JRC curves of Barton are not self-similar. In Fig. 5(a) JRC curve number 5 is shown (above) with the same scale as a fractal with the same divider dimension and amplitude as the JRC curve (bottom). To demonstrate the non self-similarity of the JRC curve a portion of Fig. 5(a) is enlarged [Fig. 5(b)]. The JRC curve is smooth while the fractal, which is self-similar, still shows

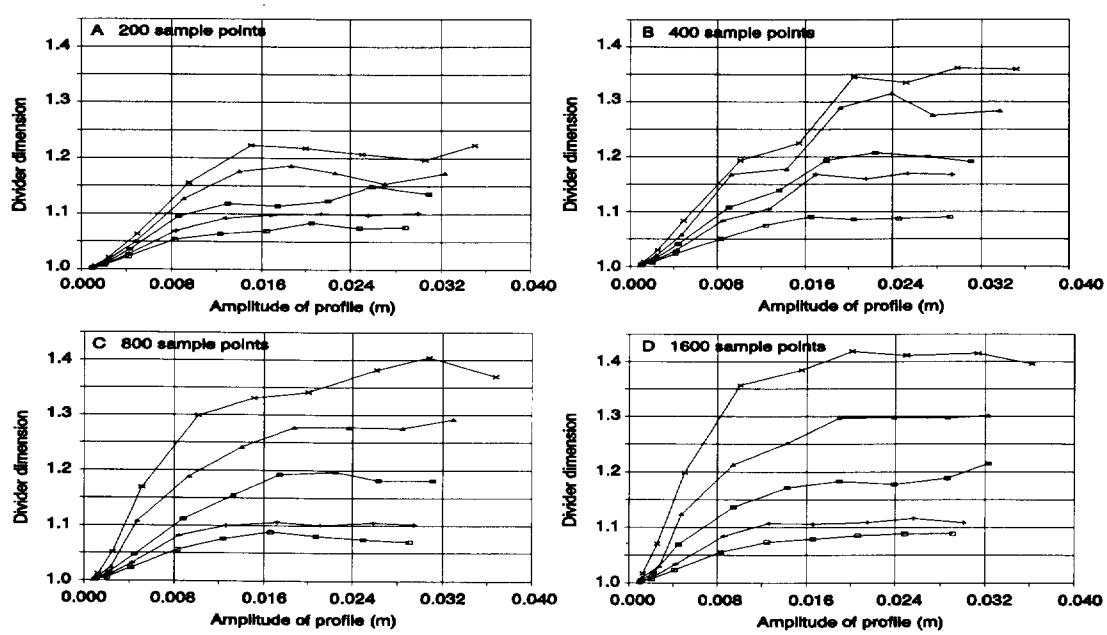

Fig. 1. Divider dimension vs amplitude for different sample point densities; fractal dimension: □ = 1.0; + = 1.1; ■ = 1.2; ▲ = 1.3; × = 1.4.

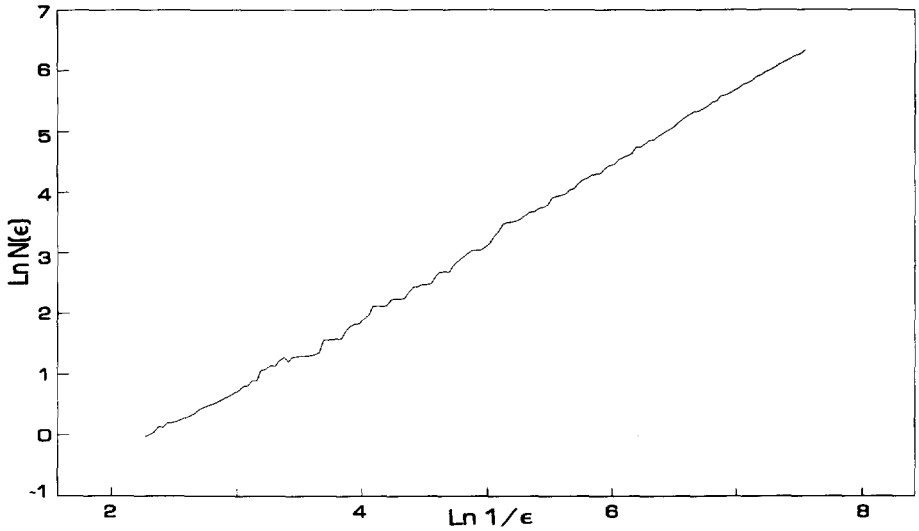

Fig. 2. Ln $N(\epsilon)$ vs ln $1/\epsilon$ with 200 sample points.

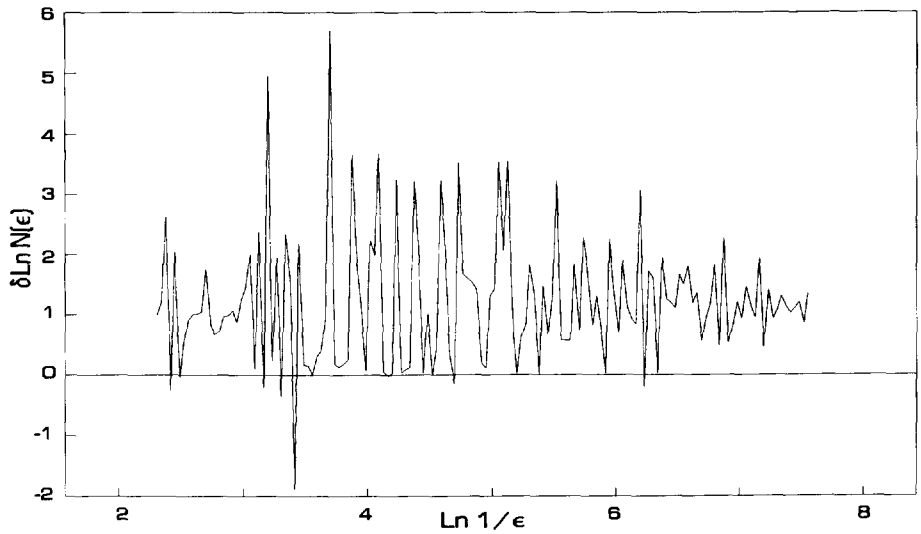

Fig. 3. δ Ln $N(\epsilon)$ vs ln $1/\epsilon$ with 200 sample points.

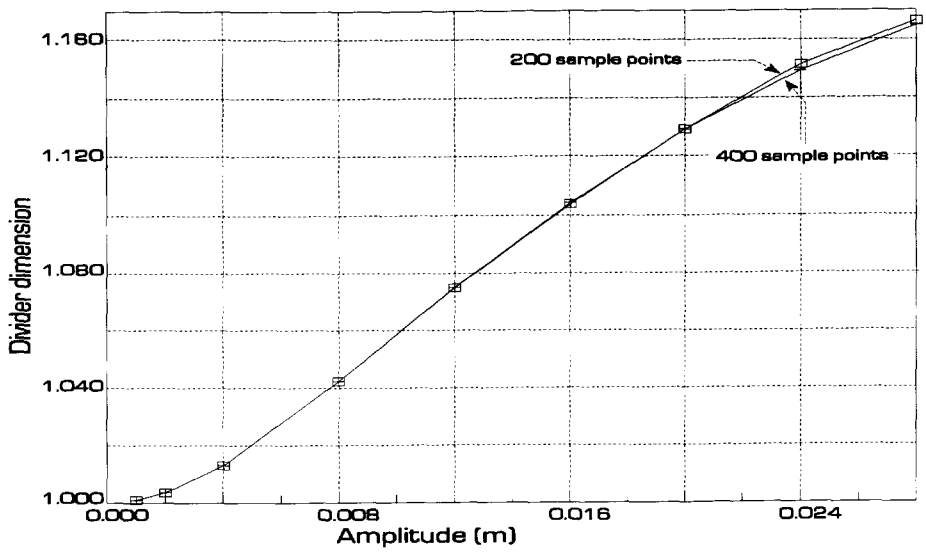

Fig. 4. Divider dimension vs amplitude for the sine function; □ = 200 sample points; + = 400 sample points.

roughness as expected for a fractal. This should lead to the conclusion that the fractal dimension of the JRC curves equals one.

Fractal dimensions for the JRC curves with the divider method can be determined although the JRC curves are not self-similar. The calculated divider dimension might, however, not describe the fractal behaviour of the curve but is likely to result from the mathematical effects mentioned before. The original published JRC curves of Barton can be represented by 20–50 data points depending on the profile number. This amount of points with linear pieces in between the data points is enough to reconstruct the original published profiles. With this amount of points no difference can visually be observed between the original and the reconstructed profiles. The authors suppose that the fractal analyses published [7, 12, 18, 19] are done with similar numbers of data points; 20–50 data points is, however, a very low density of data points for a fractal analysis. Therefore the amount of data points describing the JRC curves is increased to 3000 points by resampling the JRC curves after using parabolic splines on the 20–50 original data points. Visually there is no difference between the original published profiles, the 20–50 point profiles and the resampled profiles (on the scale of the original publication). The relationship between the JRC number and the divider dimension of the resampled data sets (Fig. 6) is comparable with relations given in the literature by Lee [12]. However, the scatter, as a result of inaccuracy in determining the slope of $\ln N(\epsilon)$ versus $\ln 1/\epsilon$, is considerably larger.

DISCUSSION

The increase of the divider dimension with increasing amplitude of a data set (Fig. 1) is due to the better spread of data points for higher amplitudes. The divider dimension is a measure of the space filling capacity of a data set and therefore responds to the amplitude of the data set if the data set does not represent a true fractal and/or if the data set is too small to mimic a fractal. Increase of the number of data points shows an improvement for the large amplitudes. The divider dimension becomes constant at the right fractal dimension and, with more data points, the divider dimension becomes constant for a smaller amplitude. The effect can be explained as follows. For a decreasing amplitude the data set becomes

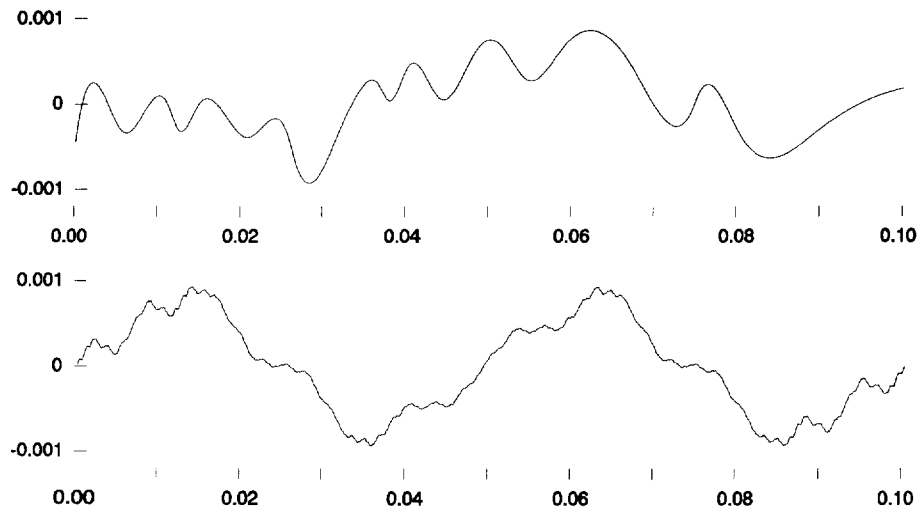

Fig. 5.(a) JRC curve No. 5; top: original; bottom: fractal generated curve (vert. scale = hor. scale = m).

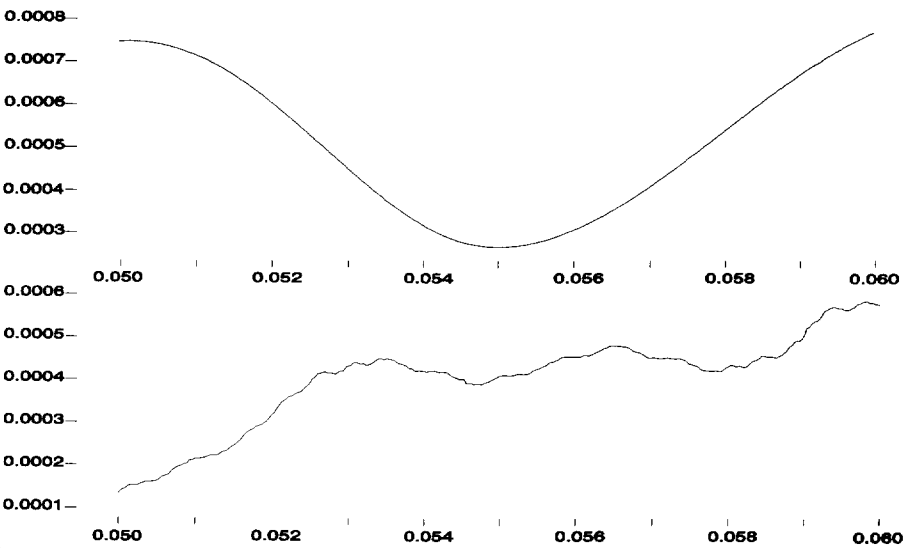

Fig. 5.(b) Detail of JRC curve No. 5; top: original; bottom: fractal generated curve (vert. scale = hor. scale = m).

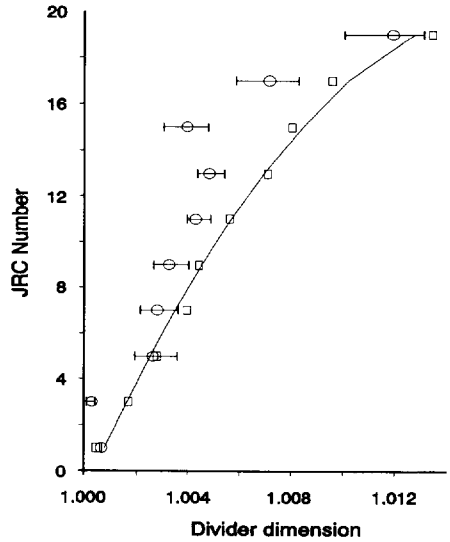

Fig. 6. JRC number vs divider dimension: ○, re-samples JRC curves with accuracy range; □, data and model from Lee [12].

more a straight, linear feature with some distortion. Limited accuracy of the method used to determine the divider dimension causes decrease of the divider dimension towards 1.0. This decrease in divider dimension is not related to the fractal dimension of the data set but is a mathematical effect as shown by the response of a sine function to the same method (Fig. 4). The sine function has a dimension of 1.0 but the calculated divider dimension increases for larger amplitudes.

A second effect is the irregularity of the tangent of $\ln N(\epsilon)$ versus $\ln 1/\epsilon$ (Fig. 2). The divider algorithm counts integer numbers of scale length ϵ. For large ϵ the variation of the tangent is larger. An increase of the amount of data points does not improve this. The linear fit is sufficiently accurate as long as the regression of $(\ln 1/\epsilon, \ln N(\epsilon))$ is done on the "correct" interval. However, no fixed rules exist. The "correct" regression interval is visually determined and is rather arbitrary. Especially near to the limits of ϵ, the curve $(\ln 1/\epsilon, \ln N(\epsilon))$ looks linear which can lead to an incorrect regression interval.

The recalculation of the divider dimensions for the JRC curves shows that a monotonic relation between the number of the JRC curve and the divider dimension of the curve might exist. However, the results are not as convincing as those published by Lee [12] and the scatter is considerably larger (Fig. 6).

The sine function results confirm the problems inherent to the determination of fractal dimension by the divider algorithm as also pointed out by Maerz and Franklin [13]. The divider dimension of a sine function with amplitude 0.004 m equals approx. 1.013 (Fig. 4). Increasing the amount of data points does not make much difference.

CONCLUSION

Contrary to the conclusions in some published literature, it is not possible, using the divider method, to differentiate between a discrete data set with a fractal dimension of 1.0 and a discrete data set with a low fractal dimension greater than 1.0. The sample density of the data set and the method of determining the fractal dimension introduce mathematical errors and/or inaccuracies that prevent the proper determination of low fractal dimensions. This causes the fractal dimension to be greater than 1.0 while the data set itself is not an image of a fractal and should have a dimension of 1.0.

A relation between roughness and the fractal dimension is therefore not straightforward and cannot be established without conditions for sampling and method of analysis.

The analyses show that any conclusion on the fact of whether fractals do or do not exist in discontinuity surfaces should be taken with care and that the characterization of a discontinuity surface as a fractal is at least very doubtful.

The relations found by other authors between divider dimension and JRC number is likely not due to the fractality of the JRC curves but rather due to the sensitivity of the divider dimension on the increasing irregularity of the curves.

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